MATH 155 - Chapter 9.8 - Power Series:
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1. **Definition: (Power Series)** If x is a variable, then an infinite series of the form

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots \end{aligned}$$

is called a **power series**. More generally, an infinite series of the form

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} a_n (x-c)^n \\ &= a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots + a_n(x-c)^n + \cdots \end{aligned}$$

is called a **power series centered at c**, where c=constant.

NOTE: To simplify the notation for power series, we need to agree that $(x-c)^0 = 1$, even if $x = c$. (ie. $0^0 = 1$)

2. **Definition: (Domain of a Series)** Let $f(x)$ be a power series centered at c. Then we way that the **domain of f(x)**,

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n,$$

is set of all x for which the power series converges. The set of all x values for which the power series converge is called the **interval of convergence.**

3. **Theorem (Convergence of a Power Series):**

Let $R > 0$ be a real number. The convergence set for a power series $\sum_{n=0}^{\infty} a_n (x-c)^n$ is always an interval of one of the following three types:

1. A single point $x = c$. In this case, the Radius of Convergence is R=0.

2. An interval $(c - R,\, c + R)$ (ie. $|x - c| < R$), plus possible one or both endpoints. In this case, the Radius of Convergence is R itself.

3. The whole real line. In this case, the Radius of Convergence is $R = \infty$.

Furthermore, a power series $\sum_{n=0}^{\infty} a_n(x-c)^n$ converges absolutely on the interior of its interval of convergence, and outside of the given interval, the power series diverges.

4. **Theorem (Properties of Functions Defined by Power Series):**

Suppose that $f(x)$ is the sum of a power series on on interval $(c-R,\, c+R)$. That is

$$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots + a_n(x-c)^n + \cdots$$

Then $f(x)$ is differentiable (hence continuous) on the interval $(c-R, c+R)$ and

1.

$$\begin{aligned} f'(x) &= \sum_{n=0}^{\infty} \frac{d}{dx}[a_n(x-c)^n] \\ &= \sum_{n=1}^{\infty} n a_n (x-c)^{n-1} \\ &= a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \cdots \end{aligned}$$

2.

$$\begin{aligned} \int f(x)\,dx &= \int \sum_{n=0}^{\infty} a_n(x-c)^n\,dx \\ &= \sum_{n=0}^{\infty} a_n \int (x-c)^n\,dx \\ &= \sum_{n=0}^{\infty} \frac{a_n(x-c)^{n+1}}{n+1} + C \\ &= C + \sum_{n=0}^{\infty} a_n \frac{(x-c)^{n+1}}{n+1} \\ &= C + a_0(x-c) + a_1\frac{(x-c)^2}{2} + a_2\frac{(x-c)^3}{3} + \cdots \end{aligned}$$

The radius of convergence of the $f'(x)$ and $\int f(x)dx$ is the same as that of the original power series. However, **the interval of convergence may be different at the end points**.